Projectile Motion Tutorial #4

– a 2D projectile motion model using numerical analysis of projectile dynamics (including aerodynamic drag)

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Introduction:

- Contrary to common belief, numerical methods are not hard use or understand. As opposed to analytical methods which are applied on large domains and in the vast majority of real life situations have no solution, the numerical methods are applied on a large number of tiny (infinitesimal) domains. Numerical analysis uses simplifying assumptions such as, linearization, short series approximations and has solutions (numerical) to most of real life problems.

- The classical motion of two planets for instance can be easily described analytically, but in the case of three planets there is no general analytical solution known to man. Numerical methods, even in the hands of a debutant, will yield solutions for systems of hundreds or thousands of planets using basic programming methods and a powerful enough computer.
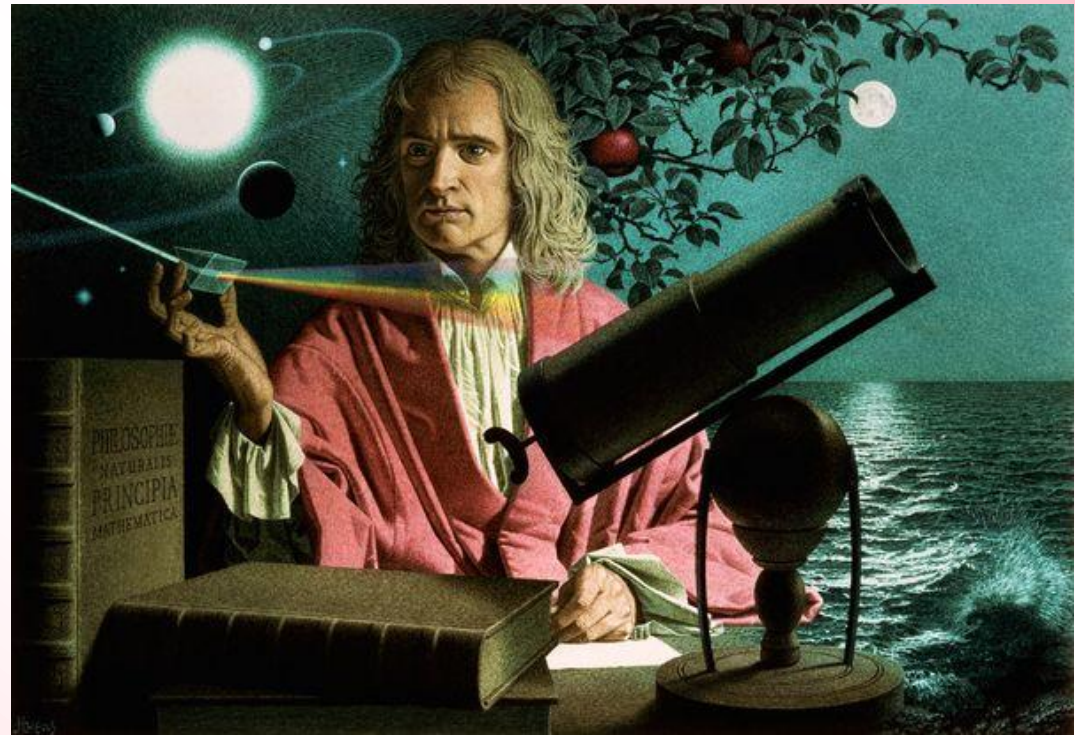
An artistic rendering of Sir Isaac Newton and his inventions

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Outline of the method:

There are two external forces applied to the projectile, the gravity and the aerodynamic drag (air resistance). The gravity is always vertical and pointing down and the drag has the same direction with the total (resultant) speed but opposite sense.

$$\vec{F}_g = -\vec{j} \cdot m \cdot g$$

$$\vec{F}_{drag} = -\frac{1}{2} \cdot \rho \cdot A \cdot Cx \cdot |v_{total}| \cdot \vec{v}_{total}$$

- The magnitude of the aerodynamic drag is half of the product of the frontal area of the object, the density of the fluid (air in this case), the square of the resultant speed and the "coefficient of drag" (also named Cx or Cd) which for a car or projectile is between 0.20 and 0.35

We will use Newton's second law to determine the acceleration:

$$\vec{F} = m \cdot \vec{a}$$

- It is a good practice to decompose the problem along the directions of the axes of coordinates, so we can use Newton's second law separately to find a_x and a_y
- The gravity has only a y component but in order to be able to determine the complete x and y components of acceleration we need to do a vector decomposition of the drag force.

- Since the drag force has the same direction and opposite sense with the total speed we can draw the following vector diagram where the pair of triangles COA and C'OA' are similar and the pair of triangles BOA and B'OA' are similar too.
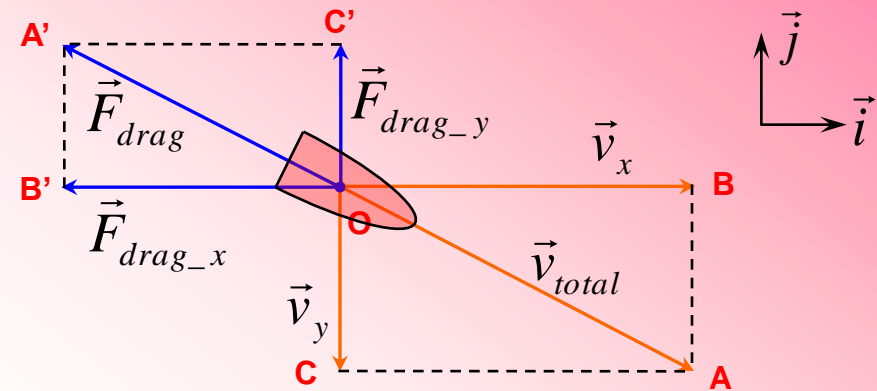
- From triangle similarities we have:

$$\begin{cases} F_{drag\_x} = -F_{drag} \cdot \dfrac{v_x}{v_{total}} \\[3em] F_{drag\_y} = -F_{drag} \cdot \dfrac{v_y}{v_{total}} \end{cases}$$

- The drag formula:

$$\vec{F}_{drag} = -\frac{1}{2} \cdot \rho \cdot A \cdot Cx \cdot |v_{total}| \cdot \vec{v}_{total}$$

- Same formula in magnitude:

$$F_{drag} = -\frac{1}{2} \cdot \rho \cdot A \cdot Cx \cdot |v_{total}^2|$$

- From Pythagoras theorem we can write:

$$|\vec{v}_{total}| = v_{total} = \sqrt{v_x^2 + v_y^2}$$

$$\begin{cases} F_{drag\_x} = -\dfrac{1}{2} \cdot \rho \cdot A \cdot Cx \cdot v_x \cdot \sqrt{v_x^2 + v_y^2} \\[3em] F_{drag\_y} = -\dfrac{1}{2} \cdot \rho \cdot A \cdot Cx \cdot v_y \cdot \sqrt{v_x^2 + v_y^2} \end{cases}$$

- Since gravity is a purely vertical force, now we have all the forces decomposed on the two axes of coordinates, x and y

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Method:

- We divide the time dimension in small "time steps" and we perform the same string of calculations during each time step. Using the speeds (v_x,v_y) and coordinates (x,y) from the previous time step we will calculate the following values (in this order):

   1) From previous speeds we calculate the x and y drag forces (we already know the gravity)

   2) From the forces we calculate the x and y accelerations using Newton's second law

   3) From accelerations and previous speeds we calculate the new x and y speeds

   4) From speeds and previous coordinates we calculate the new x and y coordinates

- The first three steps can be combined into a single step done using a larger formula
- This way we can see that there are only four formulas to calculate during each step which means that we can use only four columns of calculations to model the whole ballistics process

- We also need to know the starting speed and coordinates in order to calculate the first time step values. Values like this are called **initial conditions** and they exist and are used in any numerical method

Formulas: - This is an outline of the calculations used during each time step Δt

$$F_{current\_x} = F_{drag\_current\_x} = -\frac{1}{2} \cdot \rho \cdot A \cdot Cx \cdot v_{previous\_x} \cdot \sqrt{v_{previous\_x}^2 + v_{previous\_y}^2}$$

$$F_{current\_y} = F_{drag\_current\_y} + G = -\frac{1}{2} \cdot \rho \cdot A \cdot Cx \cdot v_{previous\_y} \cdot \sqrt{v_{previous\_x}^2 + v_{previous\_y}^2} - m \cdot g$$

1)

2)

$$a_{current\_x} = \frac{F_{current\_x}}{m}$$

$$a_{current\_y} = \frac{F_{current\_y}}{m}$$

3)

$$v_{current\_x} = v_{previous\_x} + a_{current\_x} \cdot \Delta t$$

$$v_{current\_y} = v_{previous\_y} + a_{current\_y} \cdot \Delta t$$

4)

$$x_{current} = x_{previous} + v_{current\_x} \cdot \Delta t$$

$$y_{current} = y_{previous} + v_{current\_y} \cdot \Delta t$$

To be continued...

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