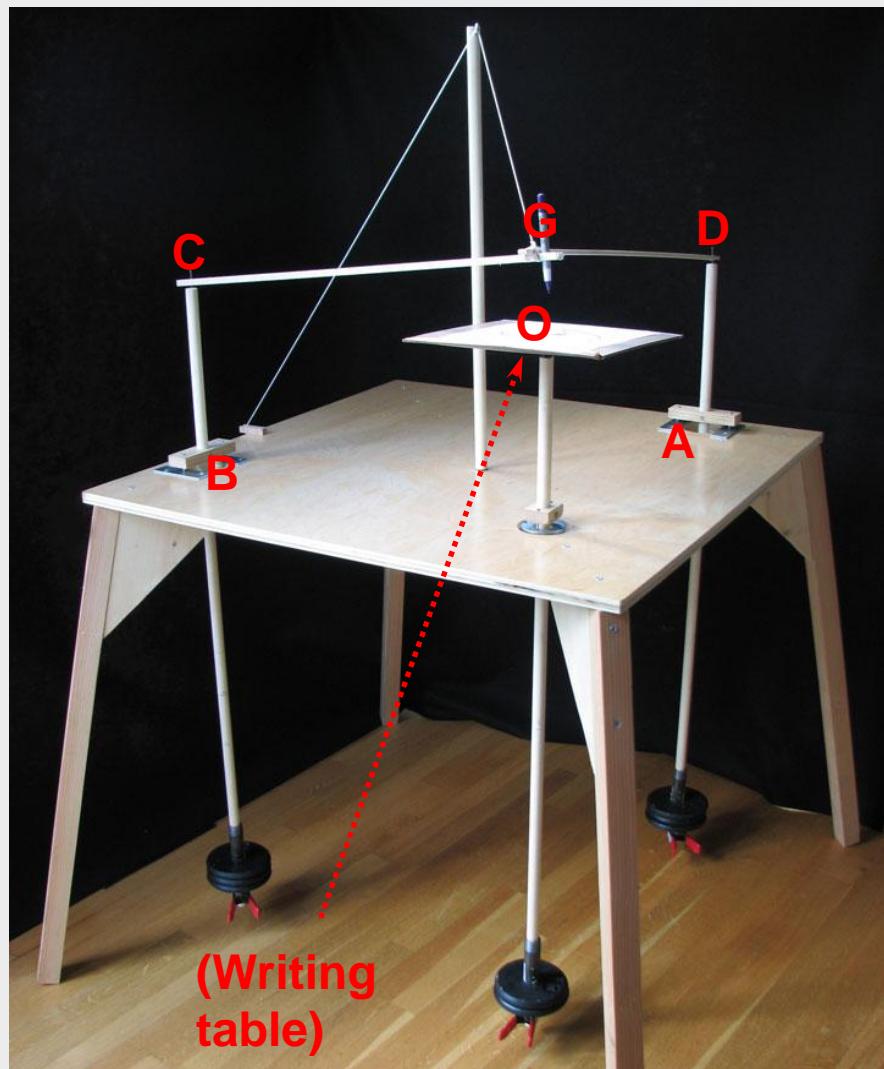


**Modeling a Three-Pendulum Harmonograph - Part #1** – by George Lungu

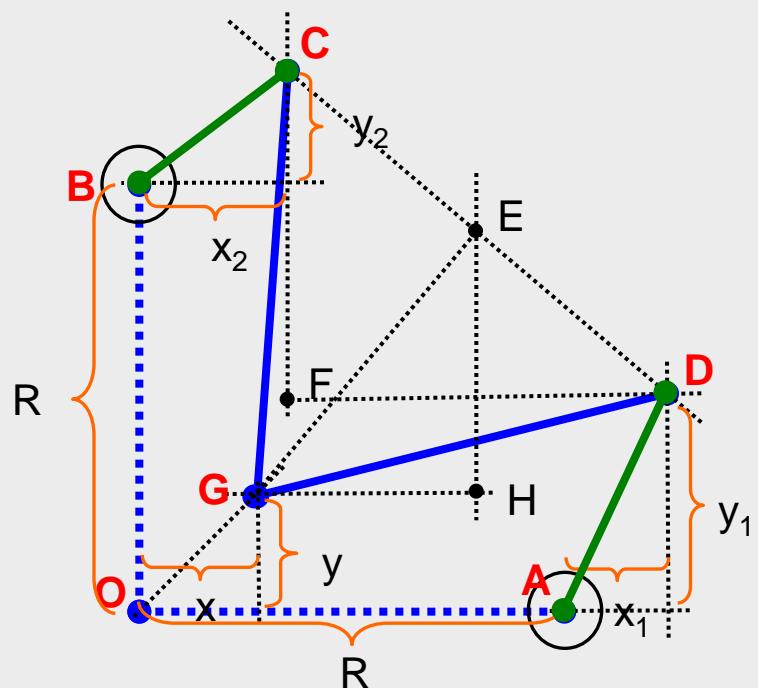
– by George Lungu

<http://excelunusual.com>



A physical implementation of a harmonograph by Karl Sims  
<http://www.karlsims.com/harmonograph/index.html>

This first part of the presentation deals with the movement equations of the three pendulums.



Let's start by modeling the movement of the points D, C and the oscillation of the table.

Let's describe the damped oscillation of the end of the **first pendulum** (point D) (you can brush-up your knowledge by searching the keyword "Lissajous"):

$$x_1 = A_1 \cdot \exp\left(\frac{-t}{t_{damping}}\right) \cdot \sin(2 \cdot \pi \cdot f_1 \cdot t)$$

$$y_1 = A_1 \cdot \exp\left(\frac{-t}{t_{damping}}\right) \cdot \sin(2 \cdot \pi \cdot f_1 \cdot t + \Delta\varphi_1)$$

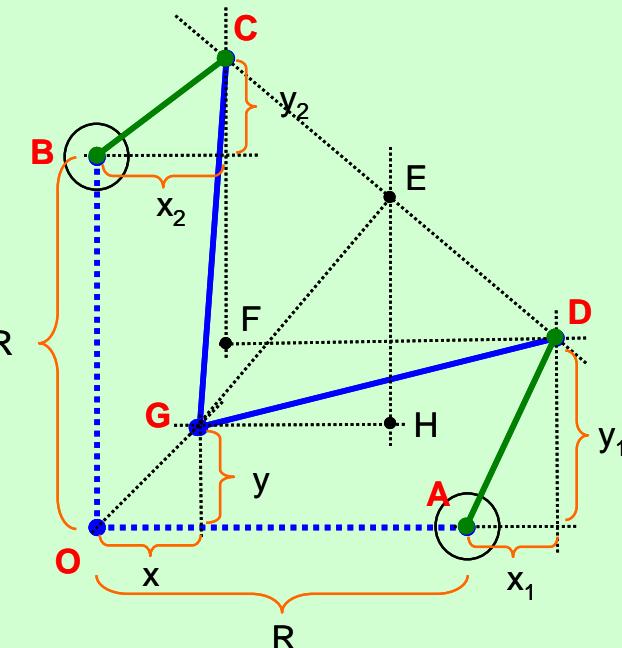
Frequency      Phase difference

**Initial amplitude**

**Damping factor**

**Damping coefficient**

**Overall Amplitude**



- The overall amplitude is not constant but will decrease exponentially in time
- The initial amplitude, damping coefficient, frequency, phase and phase difference are all adjustable
- A phase difference of  $0^\circ$  will result in a straight line oscillation, a  $90^\circ$  in a circle and in an ellipse for any angle in between

## Modeling rigid pendulum 2-D oscillations:

- Let's see some examples of how the trajectories might look like. Below there are the oscillation equations of the first pendulum

The diagram illustrates the derivation of the equations for the horizontal position  $x_1$  and vertical position  $y_1$  of a rigid pendulum over time  $t$ . The equations are:

$$x_1 = A_1 \cdot \exp\left(\frac{-t}{t_{damping}}\right) \cdot \sin(2 \cdot \pi \cdot f_1 \cdot t)$$
$$y_1 = A_1 \cdot \exp\left(\frac{-t}{t_{damping}}\right) \cdot \sin(2 \cdot \pi \cdot f_1 \cdot t + \Delta\varphi_1)$$

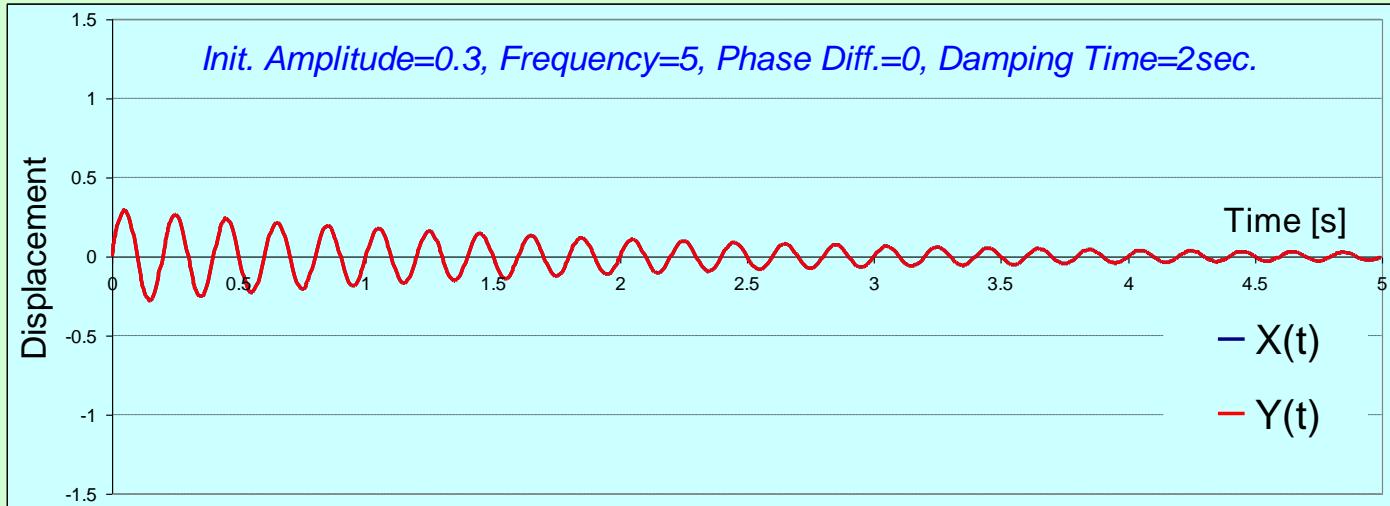
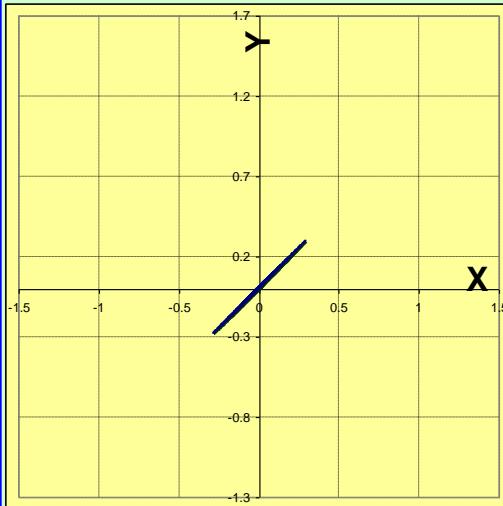
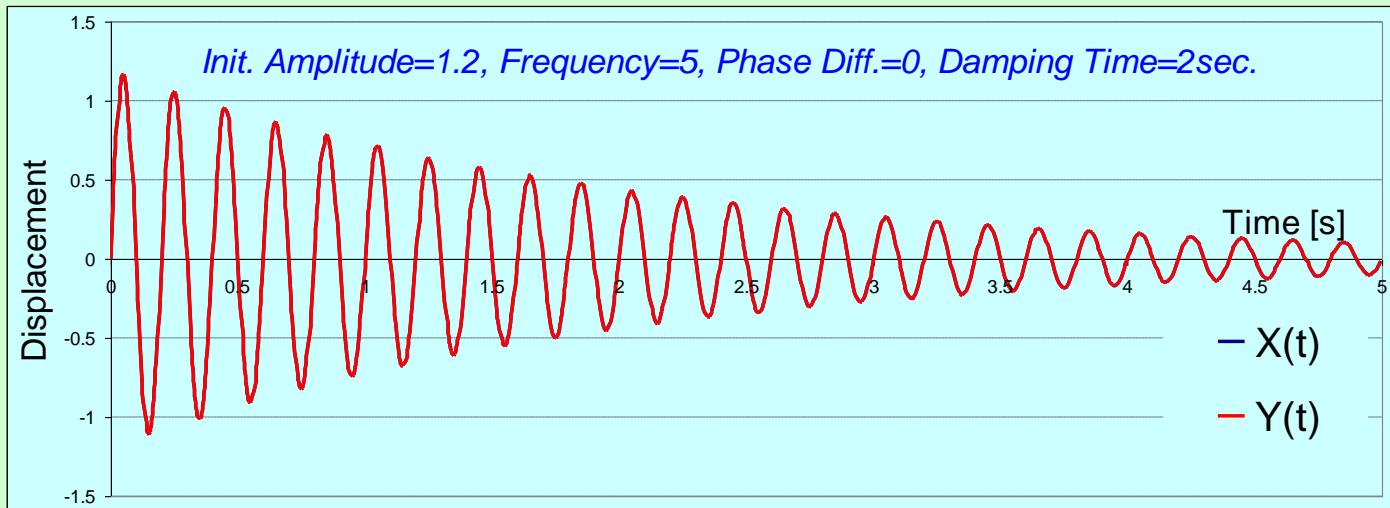
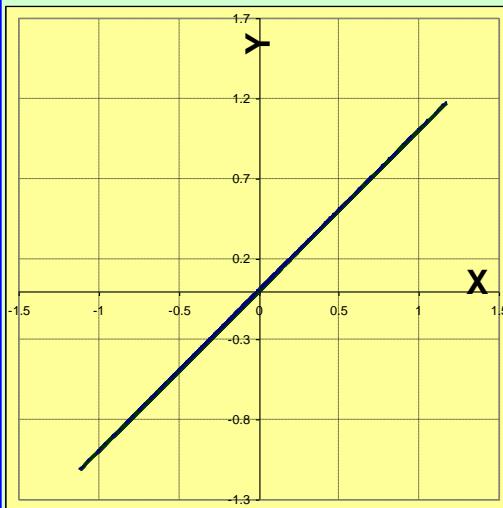
Annotations explain the components:

- Initial amplitude** ( $A_1$ ) is shown as a red circle.
- Damping factor** ( $\frac{-t}{t_{damping}}$ ) is shown as a blue dotted circle.
- Damping coefficient** ( $\frac{1}{t_{damping}}$ ) is shown as a green dotted circle.
- Overall Amplitude** is indicated by an orange bracket under the term  $\exp\left(\frac{-t}{t_{damping}}\right)$ .
- Time** ( $t$ ) is indicated by a red arrow pointing to the variable in the sine functions.
- Frequency** ( $f_1$ ) is indicated by a purple arrow pointing to the argument of the sine function in the second equation.

- Insert a worksheet named “Pendulum\_Equations” and create a table of data: X(t) and Y(t). After that, we save several parametric plots of data to see the effect of the oscillation parameters on the shape of the pendulum trajectories.

## Pendulum Parameter Effects:

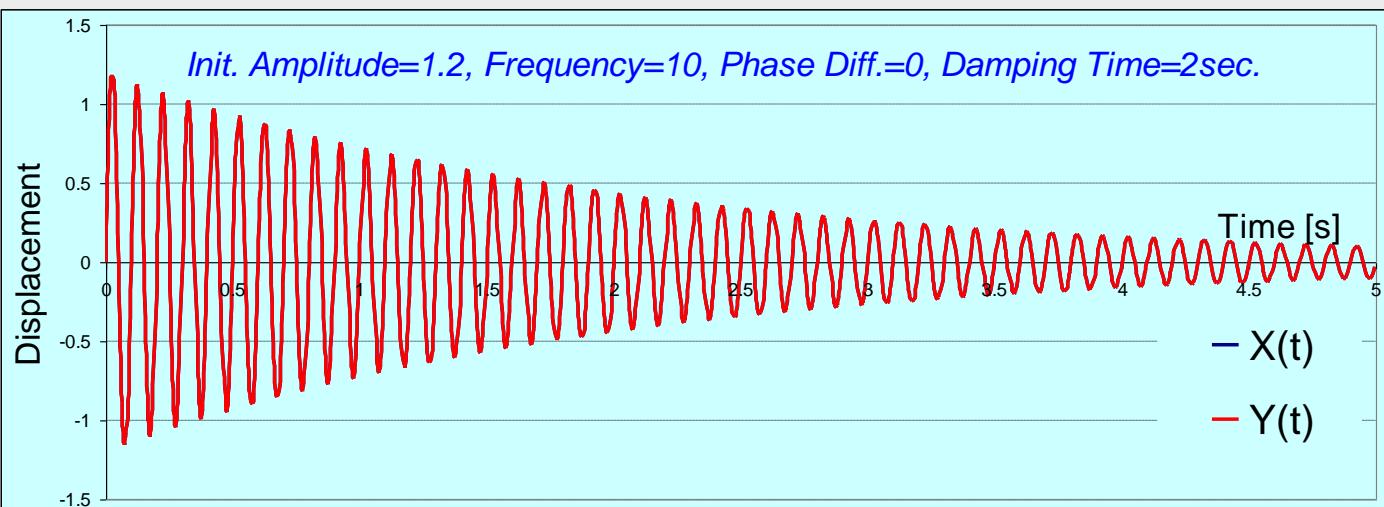
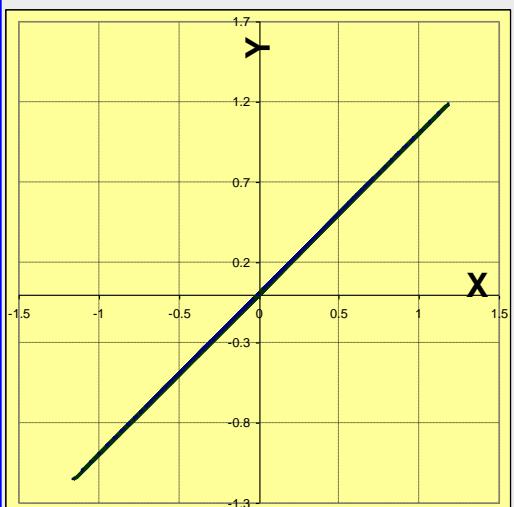
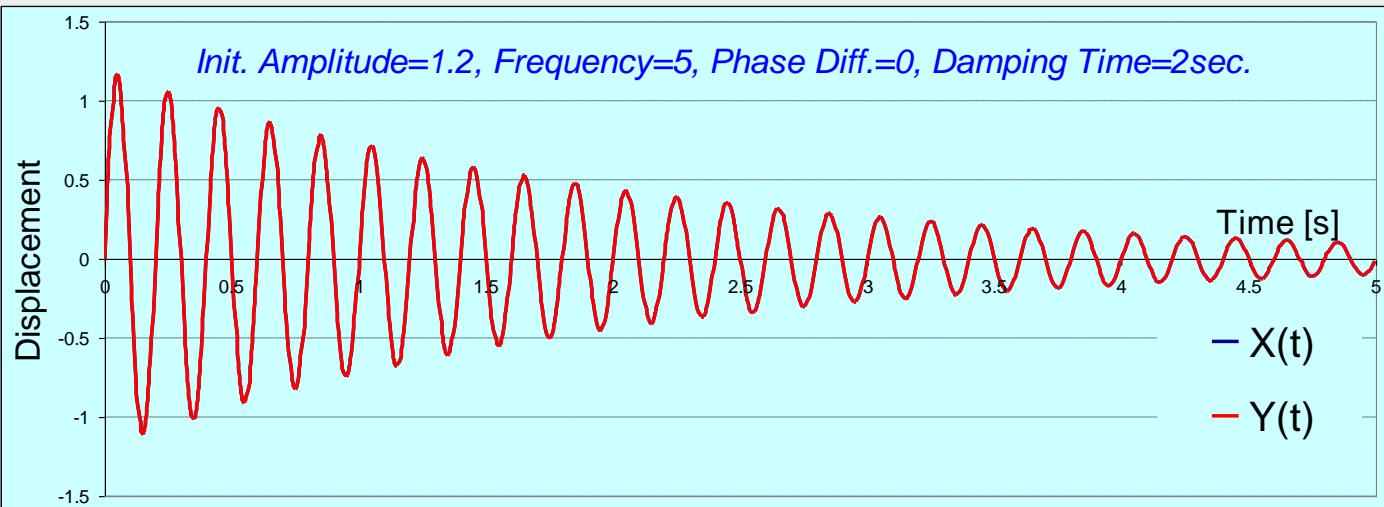
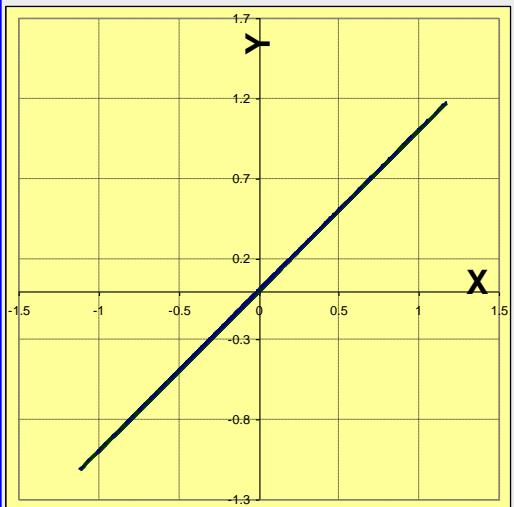
- Let's see several examples of effects of pendulum parameters on pendulum end trajectories:



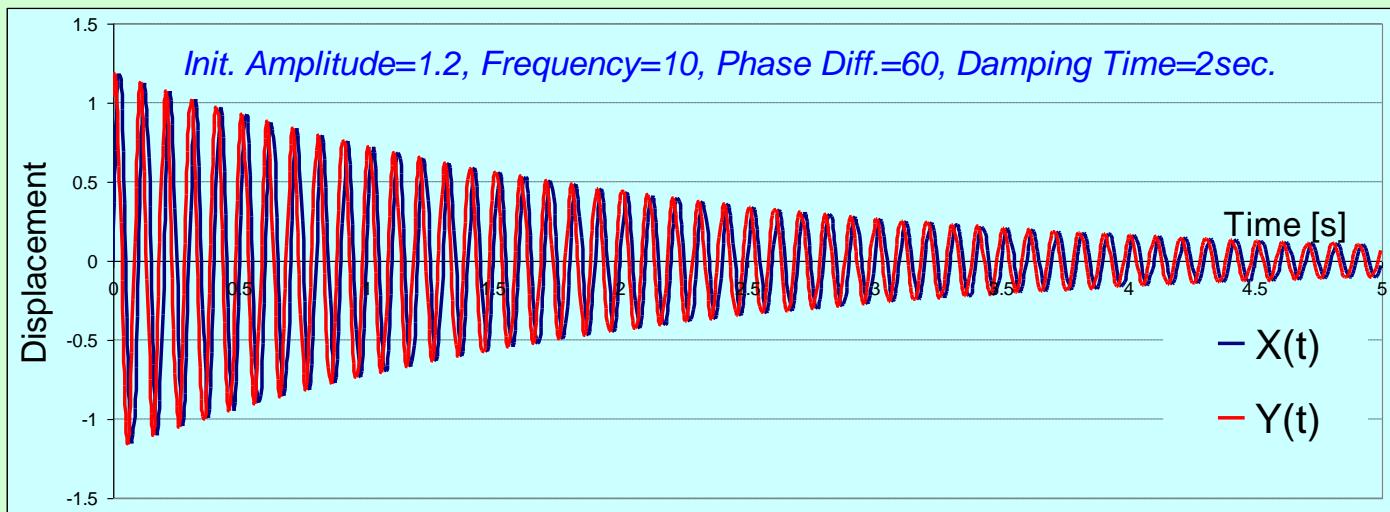
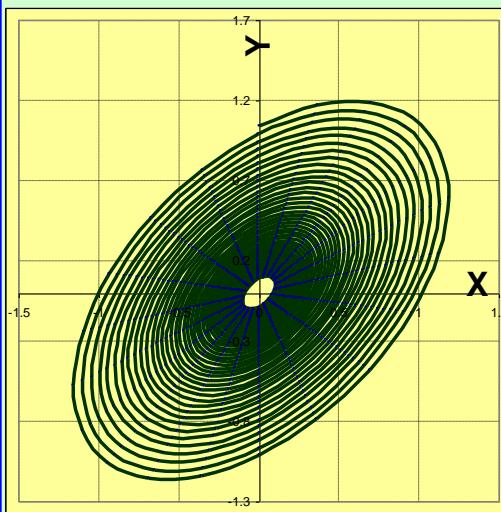
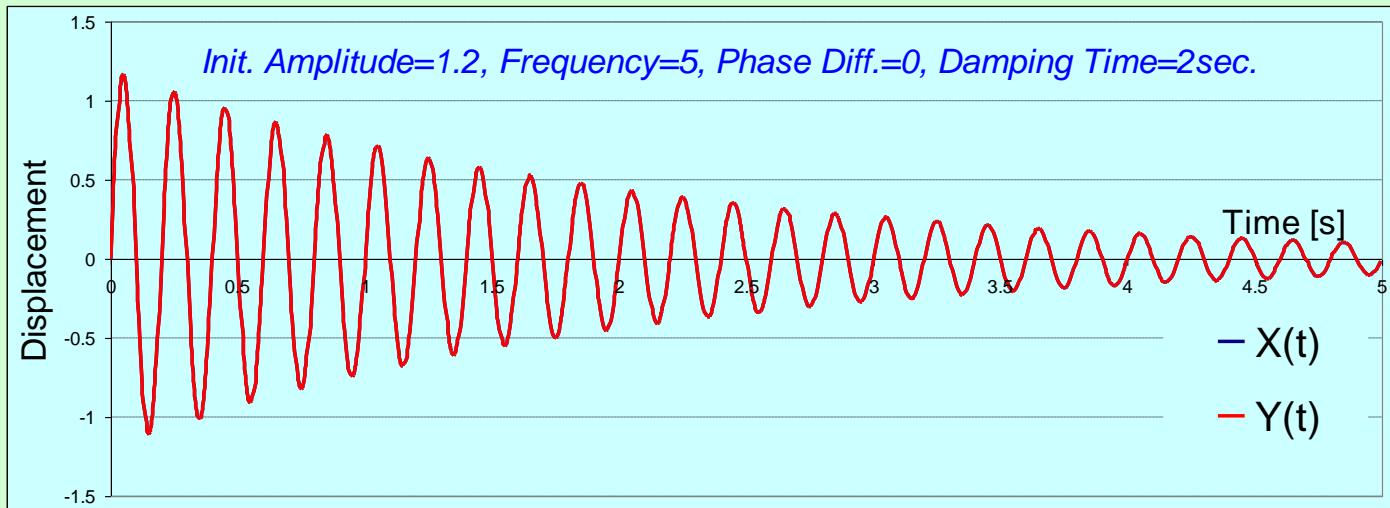
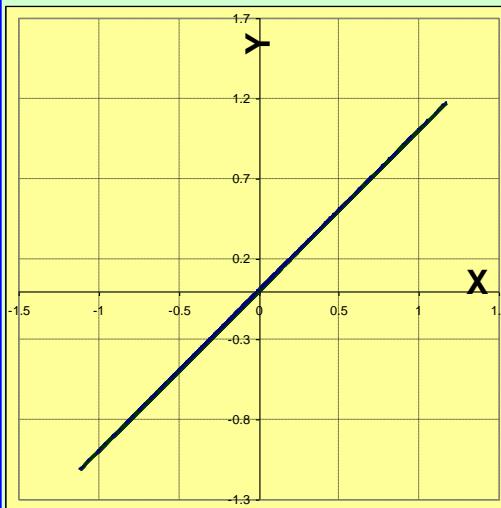
**Amplitude change**

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## Pendulum Parameter Effects:



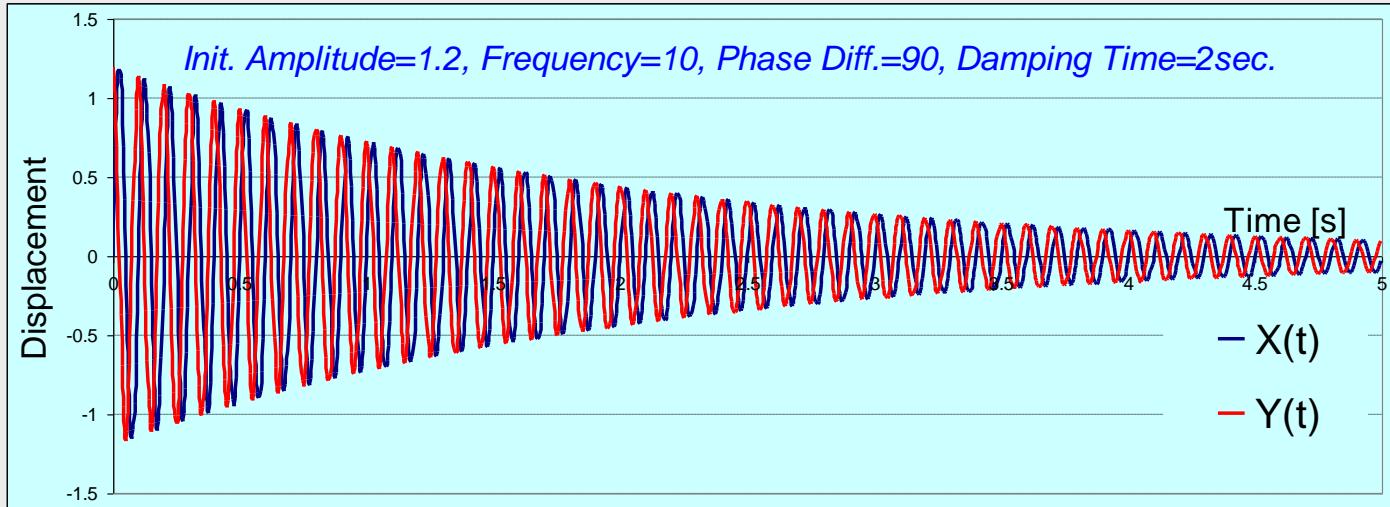
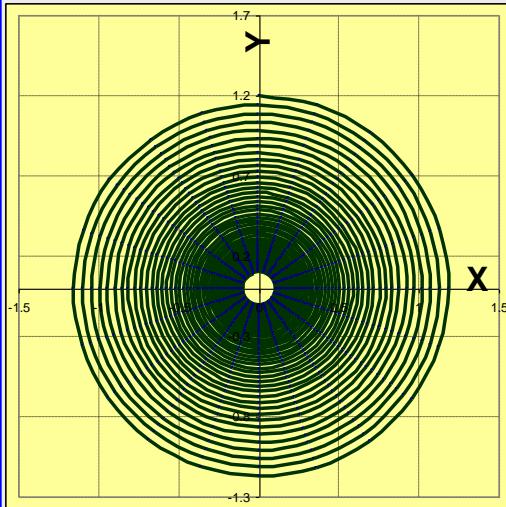
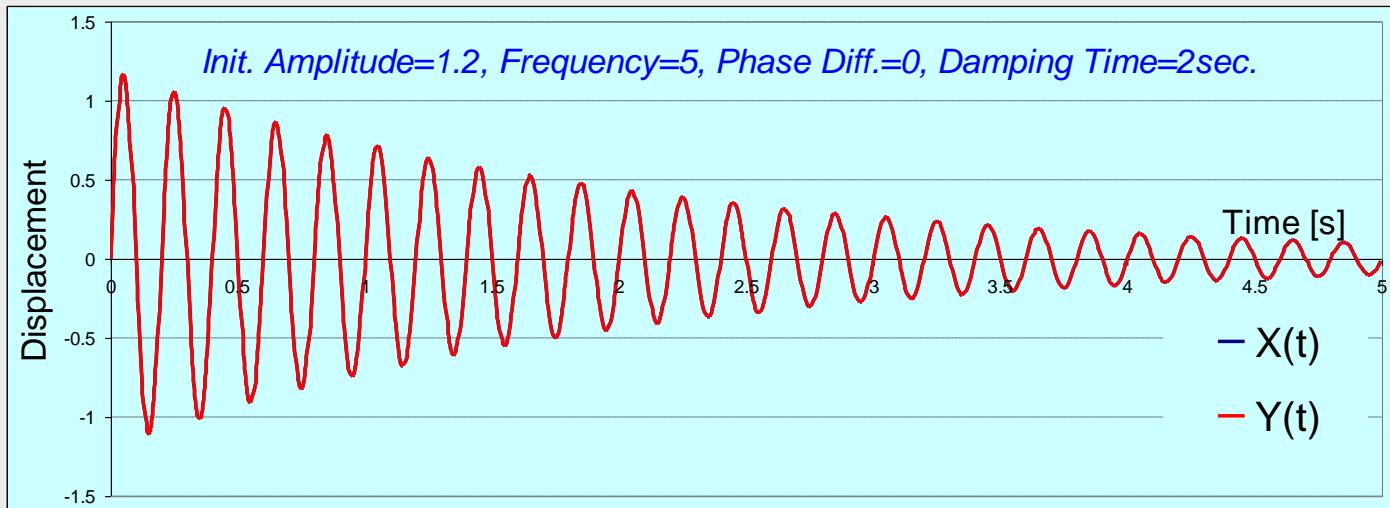
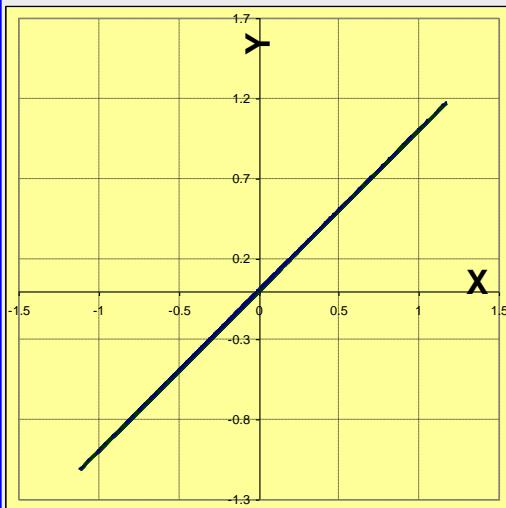
## Pendulum Parameter Effects:



*Phase Difference change*

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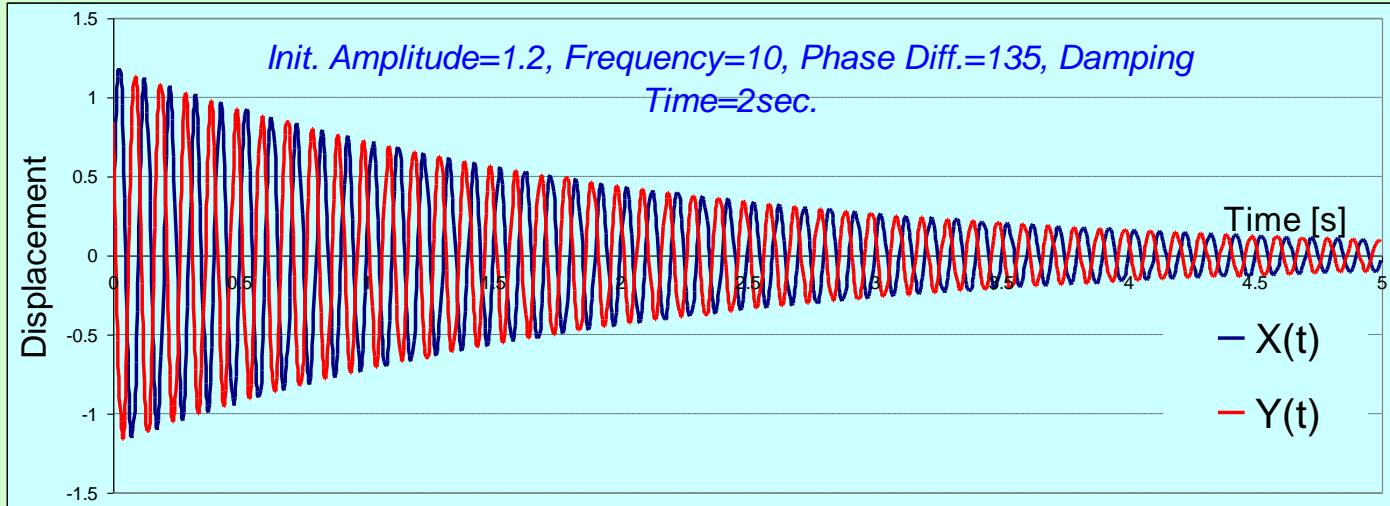
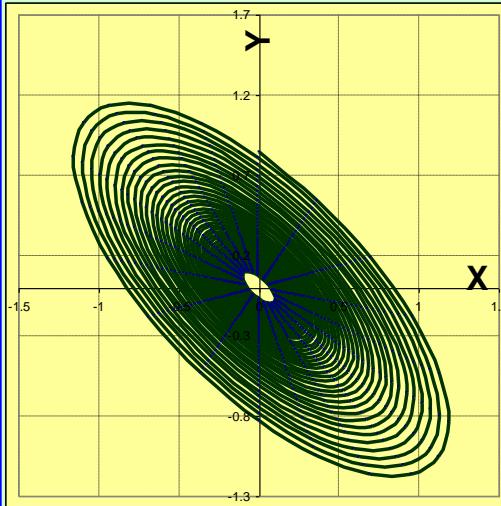
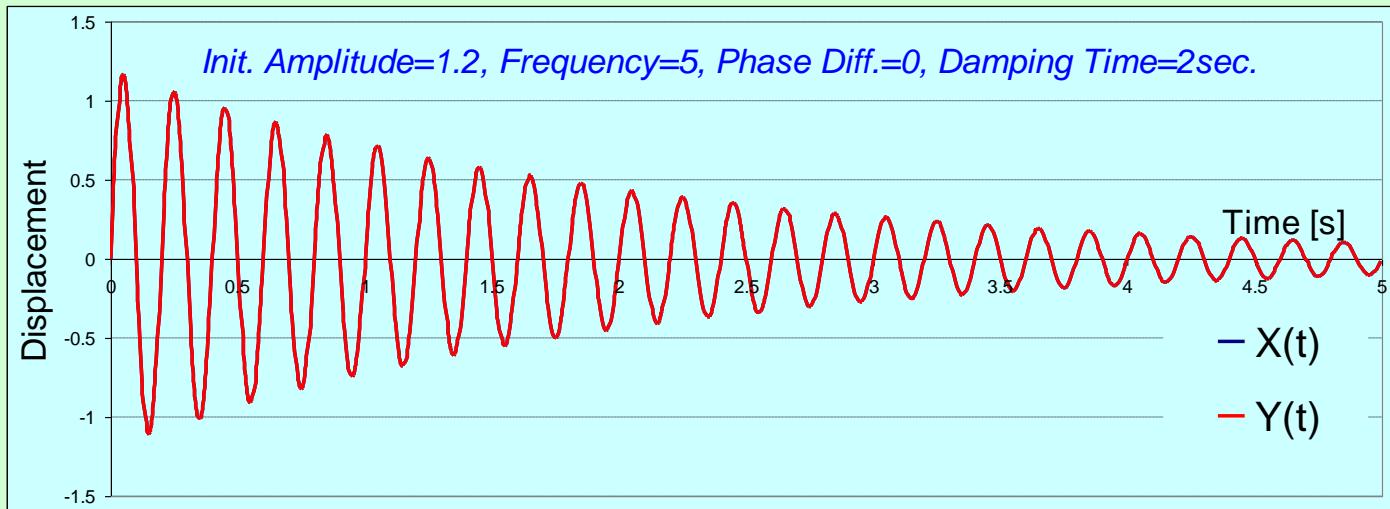
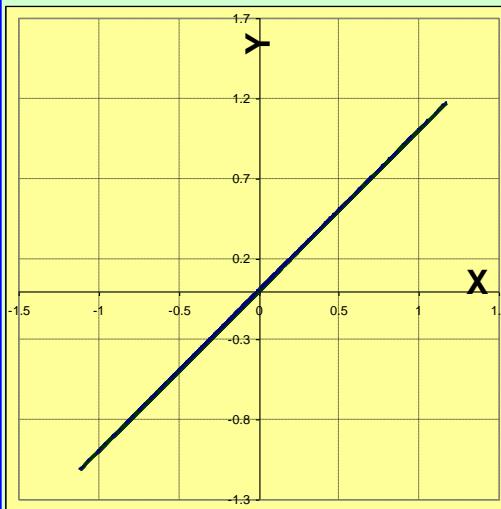
## Pendulum Parameter Effects:



*Phase Difference change*

*<excelunusual.com>*

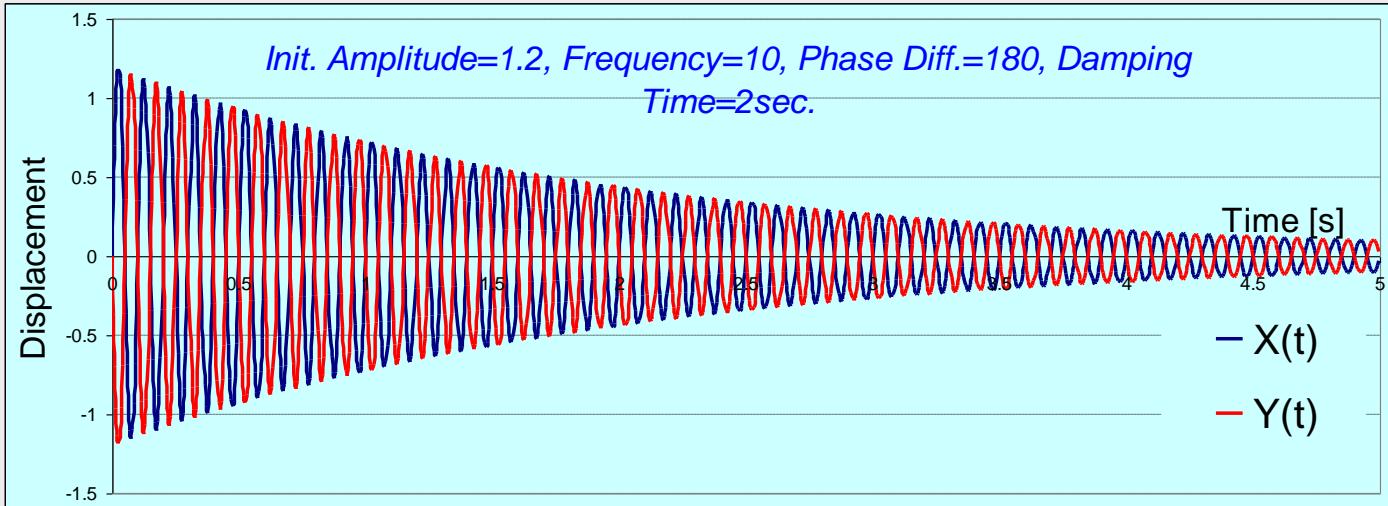
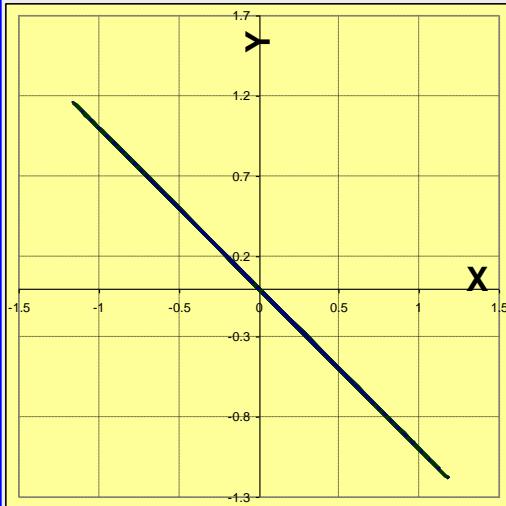
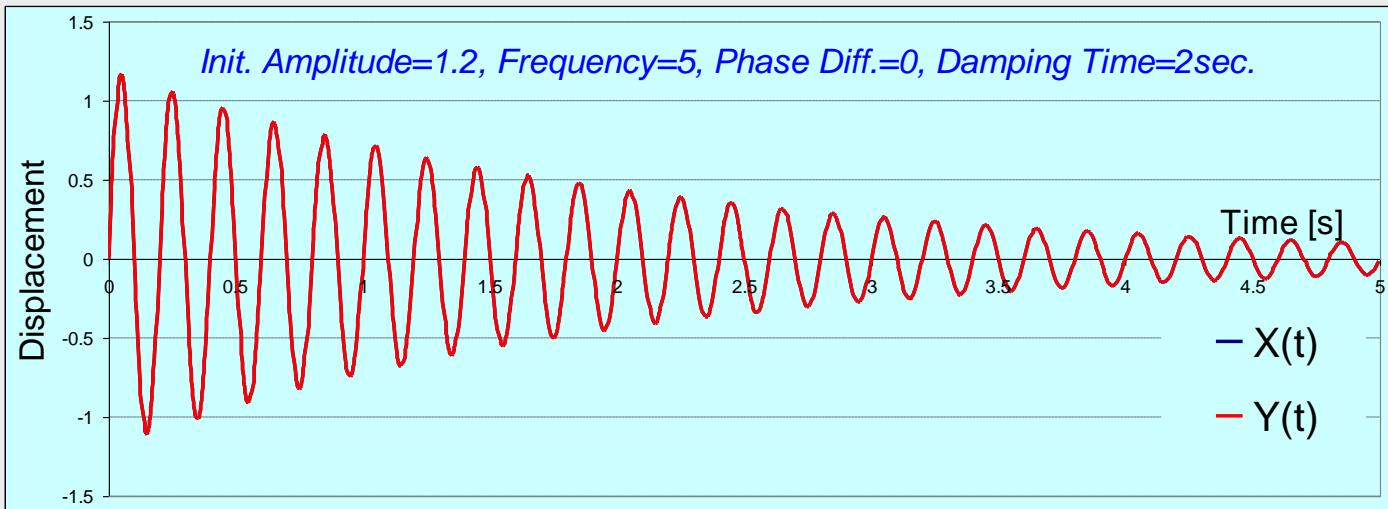
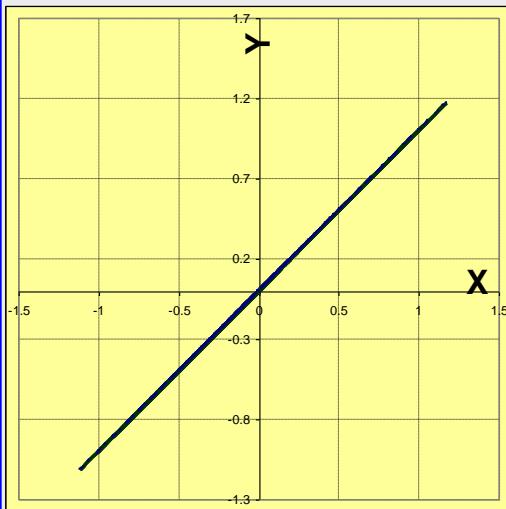
## Pendulum Parameter Effects:



*Phase Difference change*

*<excelunusual.com>*

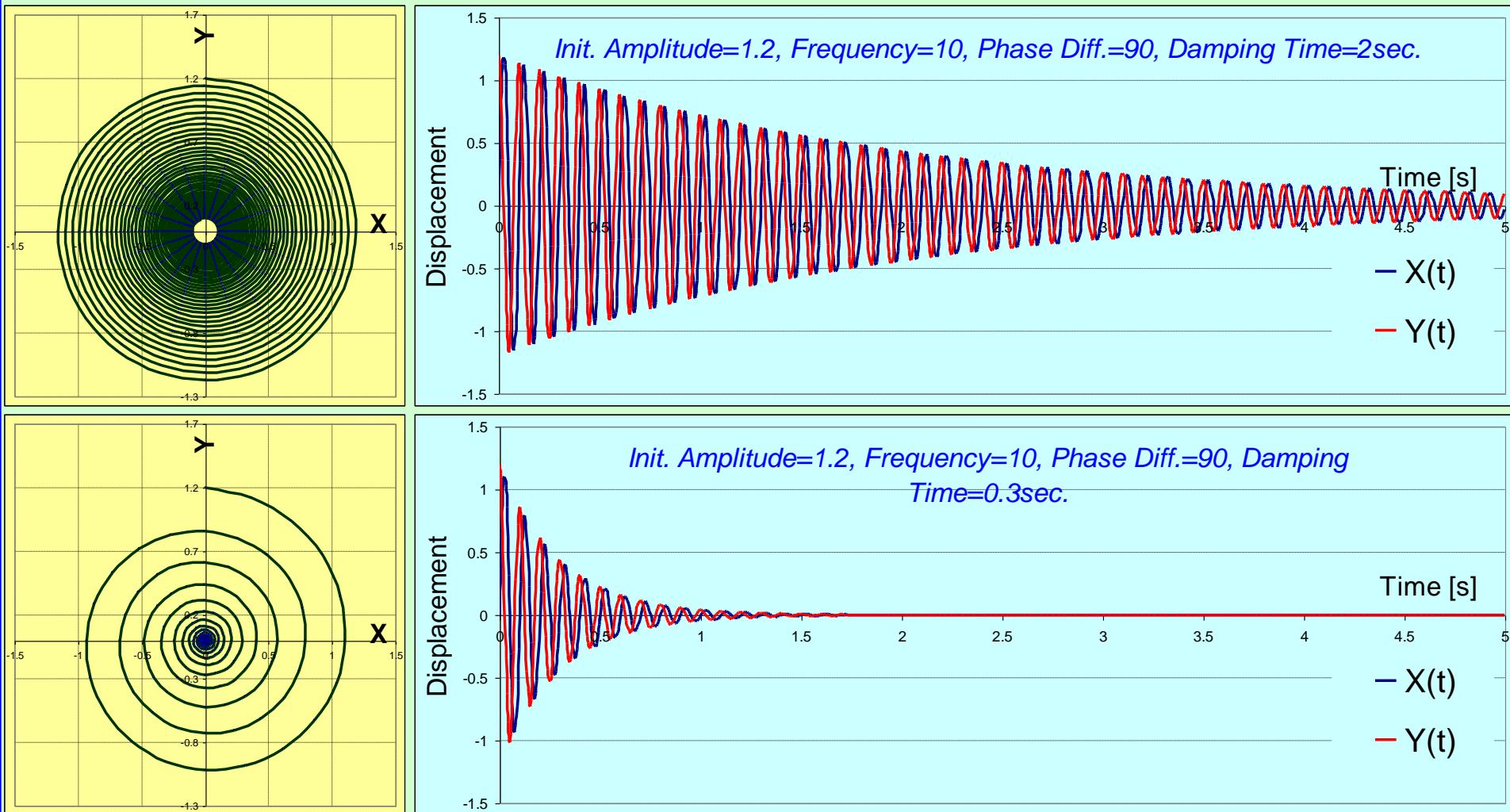
## Pendulum Parameter Effects:



*Phase Difference change*

*<excelunusual.com>*

## Pendulum Parameter Effects:



Damping Time change

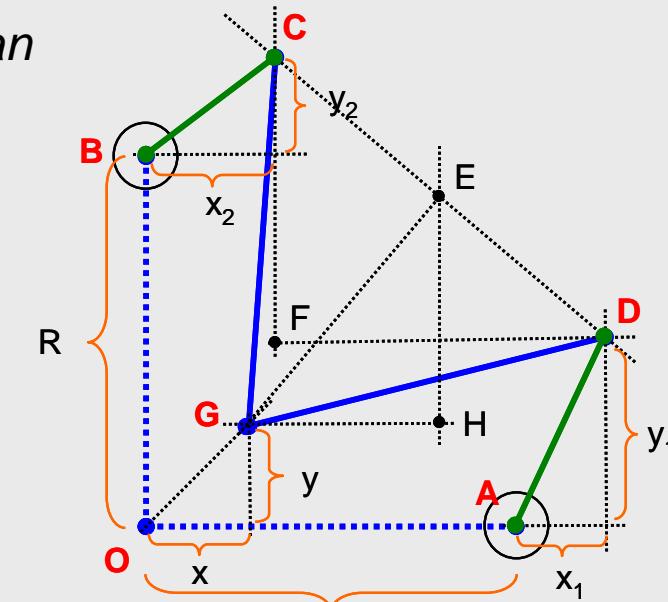
<excelunusual.com>

Coming back to the **second pendulum**, (point C) we can write the oscillation equations:

$$x_2 = A_2 \cdot \exp\left(\frac{-t}{t_{damping}}\right) \cdot \sin(2 \cdot \pi \cdot f_2 \cdot t + \varphi_2)$$

$$y_2 = A_2 \cdot \exp\left(\frac{-t}{t_{damping}}\right) \cdot \sin(2 \cdot \pi \cdot f_2 \cdot t + \varphi_2 + \Delta\varphi_2)$$

Overall Amplitude



Phase difference between the second and the first pendulum

Time

Frequency

Phase difference between x and y

- The overall amplitude is not constant but will decrease exponentially in time
- The initial amplitude, damping coefficient, frequency, phase and phase difference are all adjustable
- A phase difference of  $0^\circ$  will result in a straight line oscillation, a  $90^\circ$  in a circle and in an ellipse for any angle in between

For the **third pendulum** (the table) we can write the oscillation equations:

$$x_3 = A_3 \cdot \exp\left(\frac{-t}{t_{damping}}\right) \cdot \sin(2 \cdot \pi \cdot f_3 \cdot t + \varphi_3)$$

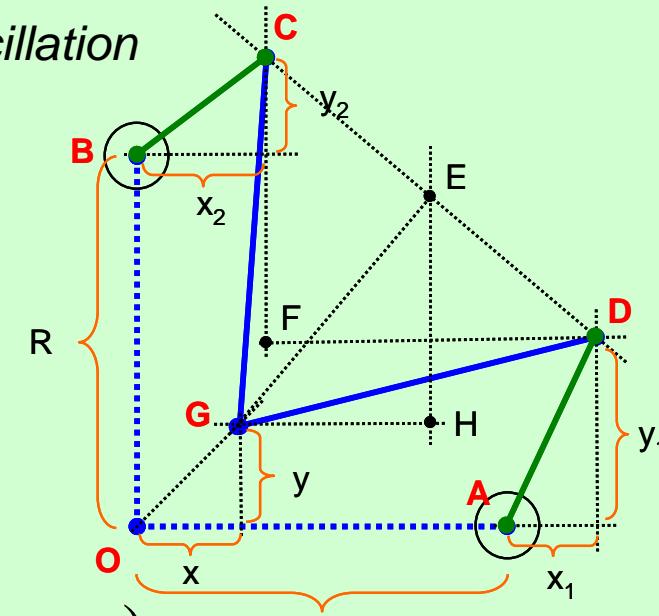
$$y_3 = A_3 \cdot \exp\left(\frac{-t}{t_{damping}}\right) \cdot \sin(2 \cdot \pi \cdot f_3 \cdot t + \varphi_3 + \Delta\varphi_3)$$

Overall Amplitude

Initial amplitude      Damping coefficient      Damping factor

Time      Frequency      Phase difference between the third and the first pendulum

Phase difference between x and y



- The overall amplitude is not constant but will decrease exponentially in time
- The initial amplitude, damping coefficient, frequency, phase and phase difference are all adjustable
- A phase difference of  $0^\circ$  will result in a straight line oscillation, a  $90^\circ$  in a circle and in an ellipse for any angle in between